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Summary. The following is a summary of the research of F. J. Narcowich and J. D. Ward supported by the Air Force during the period 6-92 through 8-95. Results dealing with center placement and stability for neural networks that employ radial basis functions (RBFs) were obtained, and led to convergent RBF identification algorithms with persistently excited regressor vector. A class of RBF-based methods that are grid-free and dimension-blind and that allow one to solve virtually any surface-fitting problem involving derivative information at scattered sites was discovered. For situations where the underlying geometries are noneuclidean—spheres or tori, for example—, basis functions analogous to RBFs were developed; these provide tools to fit not only scattered data, but also scattered derivative-data. A class of nonstationary, orthogonal, well-localized periodic scaling functions and wavelets were constructed out of these bases, so that an integrated approach to handling both the representation and analysis of periodic data, even when the data are scattered or noisy, is provided.

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"Surface Fitting with Radial Basis Functions
and Applications to Neural Networks"

by

Francis J. Narcowich and Joseph D. Ward

Final Report Period: 1 June 1992–14 August 1995

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Summary. The following is a summary of the research of F. J. Narcowich and J. D. Ward supported by the Air Force during the period 6-92 through 8-95. Results dealing with center placement and stability for neural networks that employ radial basis functions (RBFs) were obtained, and led to convergent RBF identification algorithms with persistently excited regressor vector. A class of RBF-based methods that are grid-free and dimension-blind and that allow one to solve virtually any surface-fitting problem involving derivative information at scattered sites was discovered. For situations where the underlying geometries are noneuclidean—spheres or tori, for example—, basis functions analogous to RBFs were developed; these provide tools to fit not only scattered data, but also scattered derivative-data. A class of nonstationary, orthogonal, well-localized periodic scaling functions and wavelets were constructed out of these bases, so that an integrated approach to handling both the representation and analysis of periodic data, even when the data are scattered or noisy, is provided.

I. Review of Research

The research described below was carried out during the period 1 June 1992 to the present, the early starting date stemming from a ninety-day pre-award agreement with AFOSR. The work itself is communicated in the papers and informal technical reports listed in §II. (Publications are denoted by 'P', reports by 'R', and supplementary references by 'S'.) Our research fits into four broad categories:

1. Generalized interpolation with radial basis functions (RBFs), least squares approximation and error estimates. [P1, P7, P8, P11-P13, R3, R2]
2. Periodic, spherical, and manifold analogues of radial basis functions. [P9, P10, R4]
3. Wavelets. [P2, P3, P5, P6, P14]
4. Shape preservation. [P4]

1. Generalized interpolation, least squares, error estimates. The neural networks constructed by Simmers and Southall [S1, S4] were based on linear combinations of Gaussians with variable centers. Our research dealt with the placement of these centers and the stability of the interpolation and least squares matrices associated with these radial basis functions. In [P1], we investigated the l_p bounds of the inverse of the interpolation matrices. Under the appropriate assumptions on the RBF used, we established that the interpolation matrices on l_p enjoyed the same stability properties as they did on l_2. In [P11] and [P12], we obtained estimates on spectral condition numbers for scattered-data interpolation matrices associated with a certain class of RBFs having order 0; the class includes the Gaussians. Our estimates lead to a surprising set of necessary and sufficient conditions for such condition numbers to be independent of the number of data sites. The results from our paper [P12] were used by Simmers, Southall, and O'Donnell [S4] to give a new approach to the selection of certain parameters central to their neural beamforming algorithm.

The neural beamforming algorithm mentioned above contained a novel method for representing angles. In this method, angles in an interval $[a, b]$ are represented as averages of endpoints of subintervals or "bins" formed from $[a, b]$. In private communications, Simmers and Southall remarked that the method enhanced the performance of the neural beamforming algorithm. In [R3], we give an analysis of the method, and provide some expanation for the methods effectiveness.

In [P7] and [P8], identification algorithms whose convergence and rate of convergence hinged on the regressor vector being persistently excited were discussed. We showed that if the regressor vector is constructed out of radial basis function approximants, it will be persisitently excited, provided a kind of "ergodic" condition is satisfied. Our results were closely connected with obtaining estimates for least squares stability.

One of the new aspects of RBFs uncovered in our work is the ability of RBFs to handle scattered data involving derivative information. In paper [P13], we investigated a broad class of interpolation problems, for both scalar and vector-valued multivariate functions subject to linear side-conditions, such as being divergence free, where the data are generated via integration against compactly supported distributions. We discovered a class of RBF-based methods that are grid-free and dimension-blind, and that allow one to solve virtually any surface-fitting problem involving derivative information at scattered sites in $\mathbf{R}^s$. One can even create *divergence-free* vector fields to fit data, including flux data, from magnetic fields or velocity fields for incompressible fluids. In addition, we obtained norm estimates for inverses of interpolation matrices that arise in a class of multivariate Hermite interpolation problems. In the informal report [R2], we discussed how such interpolation problems might fit into the neural beamforming algorithm.

2. Periodic, spherical, and manifold analogues of radial functions. The phases in a *phased* array antenna are angles. In particular, in the neural beamforming algorithm in [S1, S4], the data set comprises phase differences from eight antennas, and thus requires seven periodic coordinates for its description. In modern mathematical parlance, the set of all possible data is a seven dimensional torus. One way of treating this set is to regard it as imbedded in a fourteen dimensional unit cube in Euclidean space, and use standard RBF methods to fit a surface to the data. This was the method originally employed in the neural beamforming algorithm mentioned above. Unfortunately, this creates an edge effect not really present in the data set; in addition, the method requires a requires a certain amount of preprocessing before data can be fed into the neural beamforming algorithm.

The situation described above is a special case of a more general situation in which the data set resides in a noneuclidean manifold. In [P9, P10, R4], we developed tools similar to RBFs to fit not only scattered data, but also scattered derivative-data for situations where the underlying geometries are noneuclidean. Specifically, we introduced a class of positive definite kernels defined on a closed, compact, Riemannian manifold. (The m-sphere and m-torus belong to the class of such manifolds.) These kernels behave like RBFs in that they provide a grid-free method for solving uniquely a generalized version of the Hermite interpolation problem which includes interpolation for data generated by derivatives, fluxes or any other quantity one can obtain by integrating a function against a compactly supported distribution. In the cases of the sphere and the torus, the kernels $\kappa(p,q)$ have simple forms. For the sphere, $\kappa(p,q) = K(p \cdot q)$; these are called spherical basis functions (SBFs), and were introduced by Schoenberg [S3]. For the torus, $\kappa(p,q) = K(p-q)$, and we call K a periodic basis function (PBFs).

Using PBFs, Simmers, Southall, and their co-workers were able to simultaneously reduce pre-processing and remove the edge effect in their neural beamforming algoritm. PBFs and SBFs can also be used to create wavelets for the torus and the sphere. These we will describe in the next section.

3. Wavelets. Radial basis functions are well known for their ability to efficiently reconstruct functions from information at scattered-data sites, and wavelets for their ability to analyze data that arises from uniform sampling of a function. Recently, a merging of these two tools has been made. In [P2], for the case of a single real variable, we have constructed non-stationary, orthonormal, analytic wavelets generated by RBFs. These wavelets have several attractive features. Projecting a function f onto a suitable sampling space is easily done, and the associated projections approximate sufficiently smooth functions exponentially fast. Also, as in the stationary case, these orthogonal wavelets satisfy the Paley-Littlewood identity, so that perfect reconstruction of wavelet decomposition is achieved. Moreover, time frequency localization for these wavelets can be obtained and the localization is quite satisfactory in many cases.

The objective of paper [P3] was to provide a general framework for the construction of wavelets generated by translates of a given tempered distribution h, where the translates are defined by a sequence of arbitrarily spaced points. In contrast with the existing literature, where specific generating functions are considered, the approach given in [P3] is operator-theoretic in nature and relies only on certain properties of h (e.g. the order of the singularity of the Fourier transform of h at the origin), but not on a detailed knowledge of the distribution itself. This approach provides not only a unifying thread for the known results, but also a very powerful tool for analyzing almost any mathematical representation of scattered discrete (univariate) data. For example, our methods yield the first results on stability of certain wavelet bases associated with scattered data on the line under the sole assumption that the scattered data are taken on sample sites with bounded global mesh ratio. An important result of this study is an exact identification of the $L^2(\mathbf{R})$-subspaces generated by linear combinations of shifts of certain unbounded radial basis functions.

In [P5], the l_2 spectral radius of any multivariate subdivision operator S of the form

$$(S\lambda)_i = \sum_{j \in Z^s} a_{i-Mj}\lambda_j, i \in Z^s$$

where $a = \{a_i : I \in Z^s\}$ is a finitely supported mask and M is a $s \times s$ integer dilation matrix is derived. Moreover, this radius is shown to be realizeable as the largest eigenvalue (in magnitude) of an associated operator defined on a finite dimensional space whose dimension depends only on the support of the mask of S and on the ambient space $\mathbf{R}^s$. In [P6], we study the same problem for the integral operator

$$(Tf)(x) = \int_{R^s} a(x - My)f(y)dy, \quad x \in R^s.$$

In contrast to the subdivision operator we showed that the spectral radius of T on *any* $L^p(\mathbf{R}^s)$ can be computed as the spectral radius of an associated *compact* operator. In 6, we investigated a class of non-stationary wavelets generated by certain radial basis functions. For these wavelet spaces, the associated multiresolution spaces approximate sufficiently smooth functions exponentially fast. The time frequency localization can be obtained and the localization was shown to be satisfactory in many cases.

In [P14], we investigated a class of nonstationary, orthogonal, periodic scaling functions and wavelets generated by continuously differentiable periodic functions with positive Fourier coefficients; these are just PBFs for the circle. The scaling functions and wavelets presented there have a number of attractive features. The decomposition and reconstruction coefficients involve only a few terms and they be computed in terms of the FFT. To discuss the localization properites, we adapted an uncertainty relation for the circle to the case at hand, and we gave a class of wavelets that are well localized and can be as smooth as one desires. The idea is that using related bases to do both representation and analysis will provide an integrated approach to handling periodic data, even when the data are scattered or noisy. Another possible use for these wavelets is detection of damage to a neural network. Often a network will compensate for damage, and it is sometimes difficult to know when and where such damage has occurred. In the case of neural beamforming, one can use the PBF wavelets, which are naturally periodic, to continuously monitor the network to detect derivative discontinuities that would arise from damage to the network.

In addition to work on periodic wavelets, we have currently made progress in constructing wavelets capable of handling scattered data on the sphere. Preliminary results are reported in [P15].

4. Constrained approximation. The results in [P4] relate to some work of R. Shore at the Rome labs at Hanscom AFB. This paper studies best approximation in a Hilbert space X from a subset K which is the intersection of a closed convex cone C and a closed linear variety, with special emphasis on applications to the n-convex functions. Explicit formulas for computing the best approximations in this situation are given.

II. Publications, Reports, and References

Publications

[P1] B. J. C. Baxter, N. Sivakumar, and J. D. Ward, "Regarding the p-norms of radial basis interpolation matrices," *Constructive Approximation*, **10** (1994), 411-468.

[P2] C. K. Chui, J. Stöckler, and J. D. Ward, "Analytic wavelets generated by radial functions," *Advances in Computational Mathematics*, to appear.

[P3] C. K. Chui, K. Jetter, J. Stöckler, and J. D. Ward, "Wavelets for Analyzing Scattered Data: An Unbounded Operator Approach," Center for Approximation Theory Report # 350, Texas A & M University, 1994. Submitted.

[P4] F. Deutsch, V. A. Ubhaya, J. D. Ward, Y. Xu, "Constrained Best Approximation in Hilbert Space, III. Applications to n-Convex Functions," *Constructive Approximation,* to appear.

[P5] T.N.T. Goodman, C. Micchelli, and J. D. Ward, "Spectral Radius Formulas for Subdivision Operators," pgs. 335-360 in *Recent Advances in Wavelet Analysis*, ed. by L.L. Schumaker and G. Webb, Academic Press, New York, 1994.

[P6] T.N.T. Goodman, C. Micchelli, and J. D. Ward, "Spectral Radius Formulas for the Dilation-Convolution Integral Operator," Center for Approximation Theory Report # 331, Texas A&M University, 1994. Submitted.

[P7] A. J. Kurdila, F. J. Narcowich, and J. D. Ward, "Persistency of Excitation in Identification Using Radial Basis Approximants," SIAM J. Control and Optimization, **33** (1995), 625-642.

[P8] A. J. Kurdila, F. J. Narcowich, and J. D. Ward, "Persistency of Excitation, Identification, and Radial Basis Functions," in the Proceedings of the *33rd IEEE Conference on Decision and Control* held at Lake Buena Vista, Florida on December 14-16, 1994.

[P9] F.J. Narcowich, "Generalized Hermite Interpolation and Positive Definite Kernels on a Riemannian Manifold." *J. Math. Analyisis and Applications*, **190** (1995), 165-193.

[P10] F.J. Narcowich, "Grid-free methods of Hermite interpolation with RBF-like functions on a manifolds," pgs. 367-369 in *Proceedings of the 14^{th} IMACS World Congress on Computational and Applied Mathematics, I,* held at Georgia Tech on 11-15 July, 1994, ed. by W. F. Ames, IMACS, New Jersey, 1994.

[P11] F. J. Narcowich, N. Sivakumar, and J. D. Ward, "Estimates on Condition Numbers for Interpolation Matrices Associated with certain Order 0 Radial Functions," in *Advances in Computer Methods for Partial Differential Equations VII*, the Proceedings of the *Seventh IMACS International Conference on Computer methods for Partial Differential Equations,* held at Rutgers University on 22-24 June, 1992, ed. by R. Vichnevetsky, D. Knight, and G. Richter, IMACS, New Jersey, 1992.

[P12] F. J. Narcowich, N. Sivakumar, and J. D. Ward, "On Condition Numbers Associated With Radial-Function Interpolation," *J. Math. Analysis and Applications,* **186** (1994), 457-485.

[P13] F. J. Narcowich and J. D. Ward, "Generalized Hermite Interpolation via Matrix-Valued Conditionally Positive Definite Functions," *Mathematics of Computation,* **63** (1994), 661-688.

[P14] F.J. Narcowich and J. D. Ward, "Wavelets Associated with Periodic Basis Functions," *Applied and Computational Harmonic Analysis*, to appear.

[P15] F.J. Narcowich and J. D. Ward, "Nonstationary Spherical Wavelets for Scattered Data," to appear in the *Proceedings of the Eighth Texas International Conference on Approximation Theory,* held in College Station, TX, 8-12 January, 1995.

Technical Reports

[R1] "Surface Fitting with Radial Basis Functions and Applications to Neural Networks," September, 1992. (Prepared for PI retreat held at Princeton in September, 1992.)

[R2] "Appying Hermite/CPDF Surface Fitting to a Neural Network," November 12, 1992. (Informal report.)

[R3] "Binning," June 11, 1993. (Informal report.)

[R4] "RBFs for the Sphere and for the Torus," July 13, 1993 (Informal report.).

Supplementary References

[S1] T. O'Donnell, J. Simmers, D. J. Jacavanco, "Neural Beamforming for Phased Array Antennas," preprint.

[S2] E. Quak, N. Sivakumar, and J.D. Ward, *Least squares approximation with radial functions, SIAM J. Math. Anal,* **24** (1993), 1043-1066.

[S3] I. J. Schoenberg, "Positive Definite Functions on Spheres," *Duke Math. J.* **9** (1942), 96-108.

[S4] J. Simmers, H. Southall, and T. O'Donnell, "Advances in Neural Beamforming," in the *Proceedings of the 1993 Antenna Applications Symposium,* Electromagnetics Laboratory, University of Illinois, Urbana-Champaign, IL, 1993.

III. Personnel

In addition to the co-PIs F. J. Narcowich and J. D. Ward, here at Texas A&M we have interacted extensively with C. K. Chui and N. Sivakumar. Professor Chui is an internationally known member of the approximation theory community. Professor Sivakumar is a junior faculty member whose interests include RBFs. There are no graduate students currently working on the project.

IV. Activities

Laboratory Contacts. We have had consistent contact with Major (Dr.) Jeffery Simmers and Dr. Hugh Southall, who are at Hanscomb AFB in Massachusetts. During the period of this report, Major Simmers and Dr. Southall visited us here at Texas A&M University three times, in in mid-August 1992, in December 1993 and in March 1994. We paid vists to the laboratory at Hanscom AFB in July 1993, July 1994, and again in July 1995. We have exchanged numerous messages via both surface and electronic mail, and we have prepared several informal reports for them. Finally, we have also had interactions with Dr. Robert Shore, who is also at Hanscom AFB. See §I.4 above.

Other activities. Professor Narcowich gave talks related to the grant as follows. In June 1992, he spoke at the Rutgers meeting of IMACS, and in Sepetember 1992, at the Princton PI retreat. In March and October of 1993, he gave talks at the Texas A&M University Center for Approximation Theory Annual Symposium and at the *866th Meeting of the AMS* at College Station, TX, respectively. In April 1994, he again spoke at the Annual Symposium of the Center for Approximation Theory, Texas A & M University, College Station, TX. In July 1994, he gave a talk at the Fourteenth IMACS World Congress held at Georgia Tech in Atlanta, GA. In October 1994, he and J. D. Ward, gave an invited talk at the special session on Wavelet Galerkin Methods in Computational Mechanics at Society of Engineering Science 31^{st} Annual Technical Meeting held at Texas A&M University. In December 1994, he, A. J. Kurdila, and J. D. Ward contributed a talk to the 33^{rd} IEEE Conference on Decision and Control held at Lake Buena Vista, Florida. In January 1995, he gave three talks, joint with J. D. Ward and P. W Smith, at the International Conference on Approximation Theory held in College Station, TX.

In addition to the five joint talks mentioned above, Professor J. D. Ward has presented a number of talks on topics related to RBFs and RBF networks. In December 1992, he gave a talk at the IBM Watson Research Center in Yorktown Height. In January 1993, he visited Duisburg, Germany on NATO funds. In June 1993, he attended the conference on Curves and Surfaces held at Chamonix, France. He gave a talk at the session on approximation theory held at Oberwolfach in August 1993. In January 1994, he gave a talk at the winter AMS special session on wavelets held at Cincinatti, OH. In June 1994, he spoke at the special session on multivariate approximation theory of the Canadian Mathematical Society held at University of Alberta. In August 1994, he gave a talk at the summer meeting on mathematical programming held at the University of Michigan-Ann Arbor. In addition, in March 1994 he went to Oberwolfach to visit colleagues in Germany. In October 1994, he again visited Duisburg, Germany on NATO funds. Finally, in May 1995, he visited Pennsylvania State University at University Park, PA on funds from Texas A& M.